Introduction to Tropical Geometry at ICERM

February 4, 2021

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Overview: 2 friends

Tropical geometry tells us how to relate these friends:

Algebraic Geometry
Algebraic Varieties
\{ x \in \mathbb{K}^n \mid f_1(x) = \cdots = f_r(x) = 0 \}

\[ \text{trap} \]

Combinatorics
Tropical varieties/polyhedral complexes

Today:

Part I: Embedded tropical geometry via curves in the plane
Part II: Abstract tropical geometry & the two friends
Geometry over Non-Archimedean fields

Tropical Geometry deals with varieties over Non-Archimedean fields. These fields have a norm that behaves very differently from the Archimedean norm on \( \mathbb{C} \).

**Definition.** \((K, |·|)\) is an Archimedean field if it satisfies the Archimedean Axiom:

for any \( x \in K^* \), there is an \( n \in \mathbb{N} \) such that \( |nx| > 1 \).

This axiom feels natural and familiar – but \( \mathbb{R} \) and \( \mathbb{C} \) are the only complete Archimedean fields (Ostrowski's theorem).

A non-Archimedean field \( K \) is one with a norm which fails this axiom.

It comes with a function called the valuation

\[ \text{val}_K : K \to \mathbb{R} \cup \{\infty\} \]

\[ a \mapsto -\log(|a|) \quad a \neq 0, \quad 0 \to \infty \]

Example. The trivial valuation on any \( K \) is: \( 0 \mapsto \infty, \quad K^* \mapsto 0 \)

Example. The Puiseux series \( \{C(t)\} \) is:

\[ \left\{ C(t) = c_1 t^{a_1} + c_2 t^{a_2} + \ldots \mid c_i \neq 0, \ c_i \in \mathbb{C} \right\} \cup \{0\} \]

**norm:** \( |c(t)| = (\tau e)^{a_1} \)

**val:** \( \text{val}_K (c(t)) = -\log(|c(t)|) = -\log((\tau e)^{a_1}) \)

\*alg. closed\*
Embedded tropicalization

How to find the embedded tropicalization of a hypersurface over a non-Archimedean field.

**Definition.** Given a Laurent polynomial

\[ f(x) = \sum_{a \in \mathbb{Z}^n} c_a x^a \quad \in \quad K[x^{t_1}, \ldots, x^{t_n}] \]

its **tropicalization** \( \text{trop}(f) : \mathbb{R}^n \to \mathbb{R} \) is

\[ \text{trop}(f)(x) = \min_{a \in \mathbb{Z}^n} (\text{val}_K(c_a) + a \cdot x) \]

Just as we can associate a variety to \( f \), which would be a hypersurface in \((k^*)^n\), we will associate a tropical variety to \( \text{trop}(f) \).

**Definition.** The tropical hypersurface \( \text{trop}(V(f)) \) is the set

\[ \{ x \in \mathbb{R}^n \mid \text{the minimum in } \text{trop}(f)(x) \text{ is achieved at least twice} \} \]

**Example [tropical line].** Let \( f = x + y + 1 \in \mathbb{C}[[t]] [x, y] \).

Then \( \text{trop}(f) : \mathbb{R}^2 \to \mathbb{R} \), and

\[ \text{trop}(f)(w_1, w_2) = \min \left( \text{val}_K(1) + (1,0) \cdot (w_1, w_2), \text{val}_K(1) + (0,1) \cdot (w_1, w_2), \text{val}_K(1) \right) \]

\[ = \min(w_1, w_2, 0) \]

make 3 cases:
- $w_1 \& 0$ are min: $w_1 = 0$, $w_2 \geq 0$
- $w_2 \& 0$ are min: $w_2 = 0$, $w_1 \geq 0$
- $w_1 \& w_2$ are min: $w_1 = w_2$, $w_1 \leq 0$.

\[\text{tropical line}\]

\(\text{tropical line}\)

(\text{note: this would be very inefficient if you had many terms})

**Theorem** \([\text{kapranov } \Rightarrow \text{Fundamental Thm}]\)

The set \(\text{trop}(V(f))\) is the same as

\[
\{ (\text{Val}_k(y_1), \ldots, \text{Val}_k(y_n)) \mid (y_1, \ldots, y_n) \in V(f) \}
\]

**Example** \([\text{tropical line}]\). Let \(f = x + y + 1 \in \mathbb{C}[[t]][x,y]\).

\[y = -x - 1 \Rightarrow \text{all roots are } (x, -x - 1).
\]

\[\text{Kapranov Theorem } \Rightarrow \text{trop}(V(f)) = \{ (\text{Val}_k(x), \text{Val}_k(-x-1)) \mid x \in k \}
\]

\[\begin{align*}
\text{Val}_k(x) > 0 & : \text{Val}_k(-x-1) = 0 \\
\text{Val}_k(x) = 0 & : \text{Val}_k(-x-1) \geq 0 \\
\text{Val}_k(x) < 0 & : \text{Val}_k(-x-1) = \text{Val}_k(x)
\end{align*}
\]

\[\text{Remark. What kind of object is a tropical variety?}\]

A lot can be said about its structure \([\text{Structure Theorem}]\)
Proposition [Practical method for curves in the plane by hand]

Let \( f \in K[x_1^{\pm}, \ldots, x_n^{\pm}] \). The tropical hypersurface \( \text{trop}(V(f)) \) is the \((n-1)\)-skeleton of the polyhedral complex dual to the regular subdivision of the Newton polytope of \( f \) induced by the weights \( \text{val}(C_n) \).

Example. Let \( f(x,y) = t \cdot x \cdot y + x + y + t^2 \in \mathbb{C}[\{t\}][x,y] \)

1. Newton Polytope: one lattice point per monomial

\[
\begin{align*}
& y^* & t \cdot x \cdot y \\
& t^2 & x
\end{align*}
\]

2. Regular subdivision

\[
\begin{array}{c}
0 & 0 & 1 \\
2 & 0 & 0
\end{array}
\quad \mapsto \quad \mathbb{R}^3
\]

3. Dual complex (rotate by 180°)

\[
\text{trop}(f) = \min(2, x+y, 1+x+y)
\]
Exercises.

Problem 1. Let $k = \mathbb{C}[t][t]$, and let
$$f(x, y) = t^2 + tx + tx^2 + tx^3 + ty + xy + tx^2y + ty^2 + txy^2 + t^3y^3$$
1. Compute $\text{trop}(f)$.
2. Using any method you like, compute $\text{trop}(V(f))$.
3. $V(f)$ is an elliptic curve. Every elliptic curve over $\mathbb{C}[t][t]$ can be re-embedded so that its equation is of the form $y^2 = x^3 + ax + b$ for $a, b \in \mathbb{C}[t][t]$ (Weierstrass form). What are all the possibilities for tropicalizations of elliptic curves in Weierstrass form?

Problem 2. Let $a \in \mathbb{C}[t][t]^*$ and $b, c \in \mathbb{C}[t][t]$.
1. Determine $\text{trop}(V(a \cdot x + b))$.
2. Determine $\text{trop}(V(a \cdot x^2 + bx + c))$.

Problem 3. How many combinatorial types of tropical quadratic curves are there? i.e., tropicalizations of
$$0 = ax^2 + bx + c + dy + exy + fy^2$$
for $a, \ldots, f \in \mathbb{C}[t][t]^*$. 
Abstract Tropicalization

From exercise 1 in the problem session, you saw that a curve can have different embedded tropicalizations depending on the embedding:

\[ 
\begin{array}{c}
  \text{Tropical} \\
  \text{Hatton} \\
\end{array}
\]

Theorem [Chan-Sturmfels] Every elliptic curve with \( v_{i,j} = 0 \) has an embedding such that the embedded tropicalization is a honeycomb.

Question. How do we associate an "intrinsic" tropical object to a curve?
Abstract Tropicalization

A curve over $\mathbb{C}[t]$ can be thought of as a family of curves depending on a parameter $t$.

Informally, we can think of this parameter as "going to zero", and when $t=0$, we observe some possibly singular behavior.

Notation.

Let $R = \{ f \in K \mid \text{val}_K(f) \geq 0 \}$.

This is a local ring with unique maximal ideal $M = \{ f \in K \mid \text{val}_K(f) > 0 \}$.

$\text{Spec}(R)$ is a topological space with 2 points:

$$\{ \bullet, \bigtriangledown \}$$

closed open
Models of Curves

More formally, we need models of curves.

Let $K$ be complete w.r.t. $\text{val}_k$ (e.g. the completion of $\mathbb{Q}_p$).

Suppose $C$ is a smooth and proper curve over $k$.

Definition. A model $\mathcal{C}$ of $C$ over $K$ is a proper and flat scheme over $R$ whose fiber over $\mathfrak{m}$ (generic fiber) is isomorphic to $C$.

$\mathcal{C}$ is called semistable if the fiber over $\mathfrak{m}$ (special fiber) is reduced, has at worst nodal singularities, and every rational component has at least 2 singular points.

*combinatorial*
Remark. By the semistable reduction theorem we are always guaranteed that a semistable model for $C$ exists.

The semistable reduction theorem guarantees that we can put curves into a combinatorially tractable form.

You can think of this as a "good" embedding from the perspective of tropical geometry.

Example. Consider $y^2 = x^3 + x^2 + t^4$ over $\mathbb{C}[[t]]$.

smooth elliptic curve

$y^2 = (x+1)x^2$

Spec $R$
Dual Graphs

Let $C$ be a curve over $K$, and let $E$ be a semistable model.

The abstract tropicalization $\Gamma$ of $E$ is a metric graph with:

- vertices $\leftrightarrow$ irreducible components of the special fiber
- edges $\leftrightarrow$ nodes of the special fiber
- vertex weights $\leftrightarrow$ genus of the component
- edge lengths $\leftrightarrow$ deformation parameter at each node
  (locally: $xy-f$ for $f \in R$, val($f$) is length)

Example:

\[
\begin{array}{c}
\text{Example } y^2 = x^3 + x^2 + t^4 \\
\mathcal{C} \rightarrow \text{Graph}
\end{array}
\]
**Minimal Skeletons**

A tuple \((G=(V,E), w, l)\) is called a tropical curve.

\[ w : V \to \mathbb{N} \quad l : E \to \mathbb{R}_{\geq 0} \]

The genus of a tropical curve is \(\sum_{v \in V} w(v) + |E| - |V| + 1\).

We say two tropical curves \((G, w, l)\) are isomorphic if one can be obtained from the other by:

- graph automorphisms
- contracting weight 0 leaves
- "erasing" valence 2 weight 0 vertices
- contracting length 0 edges

**Remark.** Different semistable models for a curve \(C\) will have isomorphic tropical curves. (i.e., "tropical curve of \(C\)" is well-defined).

**Example.** The following tropical curves are isomorphic.
**Proposition.** Every tropical curve of genus $\geq 2$ has a minimal skeleton, which is a $(G=(V,E), l, w)$ with:

- no vertices of weight 0 and degree $\leq 2$
- no edges of length 0

**Example.** Here are all combinatorial types (i.e., forget $l$) of tropical curves of genus 2:

\[
\begin{array}{ccccccc}
\circ & \rightarrow & \circ & \circ & \circ & \circ & \circ & \circ \\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
2 Friends

How are the friends related?

- The abstract tropicalization of is .

- "Good" or faithful embedded tropicalizations of contain .

Remark. Today I focused on curves... but this can be done for higher dimensional varieties!!
Thank you!